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Dynamical solutions of warped six dimensional supergravity

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Abstract

We derive a new class of exact time dependent solutions in a warped six dimensional supergravity model. Under the assumptions we make for the form of the underlying moduli fields, we show that the only consistent time dependent solutions lead to all six dimensions evolving in time, implying the eventual decompactification or collapse of the extra dimensions. We also show how the dynamics affects the quantization of the deficit angle.

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I. INTRODUCTION

Six dimensional supergravity models have several interesting properties. Salam and Sezgin obtained static solutions in which the six dimensional gauged supergravity compactifies on a product spacetime of four dimensional Minkowski and a two dimensional sphere, $\mathcal{M}_4 \times S^2$ [1]. Remarkably this supergravity model admits the supersymmetric Minkowski vacuum while many other supergravity models do not. The modern interpretation of this property is that the solution of this theory is compatible with the introduction of branes into the spacetime. As with any massive defect, this then leads of course to the appearance of a deficit angle in the two internal spatial dimensions, as a gravitational response to the tensions of the branes [2, 3]. The resulting geometry looks like a rugby ball solution where the branes are located at the north and south pole of the ball. Gibbons, Guven and Pope (GGP) [4] showed that the Salam-Sezgin vacuum is in fact the unique one with a four dimensional maximal symmetry and general static solutions with an axisymmetric internal space.

The observation in Ref. [4] that the four dimensional spacetime is always Minkowski even in the presence of branes with tensions forms the basis of the interesting supersymmetric large extra dimension (SLED) scenario, a recent approach to solving the cosmological constant and dark energy problems [3]. If only for this reason, such is the prize at stake, it makes this six dimensional supergravity interesting from a cosmological point of view, although we note that cosmology in six dimensional supergravity has previously been studied in the context of Kaluza-Klein cosmology [5, 6]. One of the neatest aspects of the SLED model is that a 3-brane with any tension in six dimensional spacetime induces only the corresponding deficit angle and maintains a vanishing four dimensional cosmological constant, at least at the classical level. This feature is often referred to as a "self-tuning mechanism" of the effective four dimensional cosmological constant and would be expected to be part of the solution to the cosmological constant problem (although we still have to account for the affect of quantum corrections).

Although the SLED scenario has enjoyed a number of successes, open questions still remain. We are particularly interested in establishing whether the self-tuning mechanism really works in a time dependent evolving Universe (another attempt, see [7]). Previous authors have argued that the self-tuning of the four dimensional cosmological constant does

not work at least in non-supersymmetric six dimensional Einstein Maxwell theories [8, 9, 10, 11]. The work we present here has a number of overlaps with that of Tolley et al [12], (and more recently with that of Kobayashi and Minamitsuji [13]) who have obtained a series of solutions to the dynamical system based on an elegant scaling argument. We believe that the explicit expressions presented here of exact time-dependent solutions to the six dimensional supergravity model are given for the first time. We begin by deriving the underlying field equations in section II. This is followed in section III with a demonstration that it is impossible to have static internal spaces with a corresponding expanding external three dimensional space. We extend the analysis to a fully time dependent case in section IV and find a new class of exact solutions showing the nature of the instability and resulting evolution of the compact dimensions. Finally we conclude in section V.

II. THE BASIC FIELD EQUATIONS

Concentrating on the bosonic field contents of this model, we have the metric g_{MN}, dilaton ϕ, a $U(1)$ gauge field A_M with field strength F_{MN} and an antisymmetric tensor field B_{MN} whose corresponding field strength is expressed as

$$G_{MNP} = \partial_M B_{NP} + F_{MN} A_P + \text{cyclic permutations}. \tag{1}$$

The lagrangian density for the bosonic sector is given by

$$\mathcal{L}_{\rm SUGRA} = \frac{1}{2}R - \frac{1}{2}\partial^M\phi\partial_M\phi - \frac{e^{-2\phi}}{12}G^{MNP}G_{MNP} - \frac{e^{-\phi}}{4}F^{MN}F_{MN} - 2g^2e^{\phi}, \tag{2}$$

where g is the $U(1)$ gauge coupling [1]. Here M, N run over all the spacetime indices and we work on the two - sphere of radius r with the six dimensional (reduced) Planck scale $M_6 = 1$.

The field equations are

$$\Box\phi + \frac{e^{-2\phi}}{6}G^{MNP}G_{MNP} + \frac{e^{-\phi}}{4}F^{MN}F_{MN} - 2g^2e^{\phi} = 0, \tag{3}$$

$$D_M\left(e^{-2\phi}G^{MNP}\right) = 0, \tag{4}$$

$$D_M\left(e^{-\phi}F^{MN}\right) + e^{-2\phi}G^{MNP}F_{MP} = 0, \tag{5}$$

[1] Note that our definition of ϕ and g are different from those in GGP. They are related through $-2\phi_{\rm ours} = \phi_{\rm GGP}$ and $g^2_{\rm ours} = 2g^2_{\rm GGP}$.

$$
\begin{aligned}
&-R_{MN}+\partial_M\phi\partial_N\phi+\frac{e^{-2\phi}}{2}\left(G_M{}^{PQ}G_{NPQ}-\frac{1}{6}G^{OPQ}G_{OPQ}g_{MN}\right)\\
&+e^{-\phi}\left(F_M{}^PF_{NP}-\frac{1}{8}F^{PQ}F_{PQ}g_{MN}\right)+g^2e^{\phi}g_{MN}=0,
\end{aligned}\tag{6}
$$

and following the usual ansatz adopted for simplicity, from now on, we consider the case of a vanishing three form field strength

$$
G^{MNP}=0.\tag{7}
$$

Then, above field equations become

$$
\Box\phi+\frac{e^{-\phi}}{4}F^{MN}F_{MN}-2g^2e^{\phi}=0,\tag{8}
$$

$$
D_M\left(e^{-\phi}F^{MN}\right)=0,\tag{9}
$$

$$
-R_{MN}+\partial_M\phi\partial_N\phi+e^{-\phi}\left(F_M{}^PF_{NP}-\frac{1}{8}F^{PQ}F_{PQ}g_{MN}\right)+g^2e^{\phi}g_{MN}=0.\tag{10}
$$

The metric ansatz we adopt is

$$
\begin{aligned}
&ds^2=U(x^m,t)^2ds_4+r(t)^2ds_2^2,\\
&ds_4^2=-dt^2+\delta_{ij}dx^idx^j,\quad ds_2^2=\gamma_{mn}(x^m)dx^mdx^n,
\end{aligned}\tag{11}
$$

where i,j run over the usual three spatial indices, m,n run over the extra two spatial indices and $\gamma_{mn}(x^m)$ is an arbitrary two-dimensional metric. With this metric, the two form field strength takes the form of

$$
\begin{aligned}
&F_{\mu\nu}=F_{\mu m}=0,\\
&F_{mn}=F(t,x^m)\epsilon_{mn},
\end{aligned}\tag{12}
$$

with ϵ_{mn} being the anti-symmetric tensor. Here, the use of Greek indices denote the four spacetime coordinates (i.e. $\mu=(0,i)$). With the metric ansatz Eqn. (11), we write each component of the Einstein equations, $(0-0),(0-m),(i-j),(m-n)$ respectively as

$$
\begin{aligned}
&3\left(\frac{\partial_0U}{U}\right)_{,0}+2\left[\left(\frac{\partial_0r}{r}\right)_{,0}-\frac{\partial_0r}{r}\frac{\partial_0U}{U}+\left(\frac{\partial_0r}{r}\right)^2\right]+\partial_0\phi\partial_0\phi\\
&-\frac{2}{r^2}\gamma^{mn}\partial_mU\partial_nU-\frac{1}{2r^2}D_m\left(\gamma^{mn}\partial_nU^2\right)+\frac{e^{-\phi}}{8}F^{PQ}F_{PQ}U^2-g^2e^{\phi}U^2=0,
\end{aligned}\tag{13}
$$

$$
-\left(\frac{\partial_mU}{U}\right)_{,0}+4\left(\frac{\partial_0U}{U}\right)_{,m}-4\frac{\partial_mU}{U}\frac{\partial_0r}{r}+\partial_0\phi\partial_m\phi=0,\tag{14}
$$

$$
\begin{aligned}
&-\left(\frac{\partial_0 U}{U}\right)_{,0} - 2\left(\frac{\partial_0 U}{U}\right)^2 - 2\frac{\partial_0 U}{U}\frac{\partial_0 r}{r} \\
&+2\frac{\partial_m U}{U}\frac{\partial_n U}{U}\frac{U^2}{r^2}\gamma^{mn} + \frac{1}{2r^2}D_m(\gamma^{mn}\partial_n U^2) - \frac{e^{-\phi}}{8}F^{PQ}F_{PQ}U^2 + g^2 e^{\phi}U^2 = 0, \qquad (15)\\
&-\left[\left(\frac{\partial_0 r^2}{2U^2}\right)_{,0} + 4\frac{\partial_0 U}{U}\frac{\partial_0 r}{r}\frac{r^2}{U^2}\right]\gamma_{mn} + 4D_n\left(\frac{\partial_m U}{U}\right) + 4\frac{\partial_m U}{U}\frac{\partial_n U}{U} - R^{m'}{}_{mm'n} \\
&+\partial_m\phi\partial_n\phi + e^{-\phi}\left(F_m{}^P F_{Pn} - \frac{1}{8}F^{PQ}F_{PQ}g_{mn}\right) + g^2 e^{\phi}g_{mn} = 0. \qquad (16)
\end{aligned}
$$

A. The static Gibbons, Guven and Pope solution

Before discussing the time dependent solutions of this system, we recall the derivation of the static solution originally obtained by Gibbons, Guven and Pope (GGP) [4]. When we take $r(t) = 1$, $U(x^m, t) = W(x^m)$ and $\phi = \phi(x^m)$, the Einstein equations and the equation of motion for the dilaton ϕ reduce to

$$
\begin{aligned}
&-\frac{1}{4W^4}D_m(\gamma^{mn}\partial_n W^4) + \frac{e^{-\phi}}{8}F^{PQ}F_{PQ} - g^2 e^{\phi} = 0, \qquad (17)\\
&-4D_n\left(\frac{\partial_m W}{W}\right) - 4\frac{\partial_m W}{W}\frac{\partial_n W}{W} + R^{m'}{}_{mm'n} \\
&-\partial_m\phi\partial_n\phi - e^{\phi}\left(F_m{}^P F_{nP} - \frac{1}{8}F^{PQ}F_{PQ}\gamma_{mn}\right) - g^2 e^{\phi}\gamma_{mn} = 0, \qquad (18)
\end{aligned}
$$

and

$$
\frac{1}{W^4}D_m\left(\gamma^{mn}W^4\partial_n\frac{\phi}{2}\right) + \frac{e^{-\phi}}{8}F^{MN}F_{MN} - g^2 e^{\phi} = 0, \qquad (19)
$$

leading to the GGP solution:

$$
\phi = -\frac{1}{2}\ln W^4. \qquad (20)
$$

The equation of motion (9) for F_{mn} leads to the solution to Eqn. (12)

$$
F = -\frac{q}{2}e^{\phi}W^{-4} = -\frac{q}{2}W^{-6}, \qquad (21)
$$

where we have used Eq. (20). Note that q can be interpreted as a magnetic charge.

It will prove useful to recall the explicit solution for ϕ and γ_{mn} as presented by GGP [4, 14]

$$
\begin{aligned}
&\gamma_{mn} = diag(\gamma_{rr}, \gamma_{\psi,\psi}), \\
&(\gamma_{rr}, \gamma_{\psi\psi}) = \left(\frac{e^{-\phi}}{f_0^2}, \frac{e^{-\phi}r^2}{f_1^2}\right), \qquad (22)\\
&e^{2\phi} = \frac{f_0}{f_1}, \qquad (23)
\end{aligned}
$$

with

$$f_0 \equiv 1 + \frac{r^2}{r_0^2}, \quad f_1 = 1 + \frac{r^2}{r_1^2}, \tag{24}$$

$$r_0^2 = \frac{1}{g^2}, \quad r_1^2 = \frac{8}{q^2}. \tag{25}$$

In the following sections we begin to explore the dynamical equations by allowing the scale factors and the fields to become time dependent.

III. ONLY ONE EVOLVING SPACE IS NOT A SOLUTION

A. Time dependent U

Ideally what we want to obtain is a solution which describes the expansion of our three space dimensions with a static extra dimensional space. As a first step towards obtaining it, we make the ansatz of a static internal space $r = 1$, and a static dilaton $\phi = \phi(x^m)$. This combination makes sense in that the relation $r^2 \propto e^{-\phi}$ has previously been obtained in [3] hence the dilaton would be static if r is static. In fact, in the following subsection, we will show that allowing for a time dependence of ϕ does not improve the possibility of obtaining static r solutions. The field equations are reduced to

$$\frac{3}{U^2}\left(\frac{\partial_0 U}{U}\right)_{,0} - \frac{1}{4U^4}D_m(\gamma^{mn}\partial_n U^4) + \frac{e^{-\phi}}{8}F^{PQ}F_{PQ} - g^2 e^{\phi} = 0, \tag{26}$$

$$\left(\frac{\partial_m U}{U}\right)_{,0} - 4\left(\frac{\partial_0 U}{U}\right)_{,m} = 0, \tag{27}$$

$$-\frac{1}{U^2}\left(\frac{\partial_0 U}{U}\right)_{,0} - \left(\frac{\partial_0 U}{U}\right)^2 \frac{2}{U^2} + \frac{1}{4U^4}D_m(\gamma^{mn}\partial_n U^4) - \frac{e^{-\phi}}{8}F^{PQ}F_{PQ} + g^2 e^{\phi} = 0, \tag{28}$$

$$\begin{aligned}&4D_n\left(\frac{\partial_m U}{U}\right) + 4\frac{\partial_m U}{U}\frac{\partial_n U}{U} - R^{m'}{}_{mm'n} \\ &+\partial_m\phi\partial_n\phi + e^{-\phi}\left(F_m{}^P F_{Pn} - \frac{1}{8}F^{PQ}F_{PQ}\gamma_{mn}\right) + g^2 e^{\phi}\gamma_{mn} = 0,\end{aligned} \tag{29}$$

$$\frac{1}{U^4}D_m\left(\gamma^{mn}U^4\partial_n\frac{\phi}{2}\right) + \frac{e^{-\phi}}{8}F^{MN}F_{MN} - g^2 e^{\phi} = 0. \tag{30}$$

Although the equations look intractible, we can make progress by noting that because

$$\left(\frac{\partial_m U}{U}\right)_{,0} = \frac{\partial_0\partial_m U}{U} - \frac{\partial_0 U \partial_m U}{U^2} = \left(\frac{\partial_0 U}{U}\right)_{,m}, \tag{31}$$

then from Eq. (27),

$$\left(\frac{\partial_m U}{U}\right)_{,0} = \left(\frac{\partial_0 U}{U}\right)_{,m} = 0. \tag{32}$$

The general solution of U is therefore given by $\ln U = \ln a(t) + \ln W(x^m)$ where $a(t)$ and $W(x^m)$ are integration functions. Thus, we find that U has to take the separable form of $U = a(t)W(x^m)$. The field equations (26) and (28) can then be reduced to

$$C(x^m) - \frac{1}{4W^4} D_m(\gamma^{mn}\partial_n W^4) + \frac{e^{-\phi}}{8} F^{PQ} F_{PQ} - g^2 e^{\phi} = 0, \tag{33}$$

$$\frac{3}{U^2}\left(\frac{\partial_0 a}{a}\right)_{,0} = \frac{1}{U^2}\left(\frac{\partial_0 a}{a}\right)_{,0} + \frac{2}{U^2}\left(\frac{\partial_0 a}{a}\right)^2 = C(x^m), \tag{34}$$

which after some algebra leads to

$$a(t) = \frac{a_0}{t - t_0}, \tag{35}$$

$$C(x^m) = \frac{3}{a_0^2 W(x^m)^2}, \tag{36}$$

with a_0 and t_0 being integration constants.

However, from Eqs. (26) and (30), we also know that

$$D_m\left(\gamma^{mn} U^4 \partial_n(\ln U^4 + 2\phi)\right) - 4U^4 C(x^m) = 0, \tag{37}$$

which is conflict, for any non-vanishing U on the internal space, with the fact that the extra two dimensional space is compact. In other words, if we integrate both sides of Eq. (37) over the compact extra space, we see that the first term on the left hand side vanishes as it is a total derivative while the second term does not. Hence, there is an inconsistency and so we conclude there is no static solution for the compact space with this ansatz for U.

There is a caveat to this argument. We have implicitly assumed that the extra space is smooth. However, if we allow the extra space to be singular then it is possible that, de-Sitter type solutions may be obtained [16].

B. Time dependent U and ϕ, but static r.

We now allow for a time-dependent dilaton. The field equations are

$$3\left(\frac{\partial_0 U}{U}\right)_{,0} + \partial_0\phi\partial_0\phi - \frac{1}{4U^2} D_m\left(\gamma^{mn}\partial_n U^4\right) + \frac{e^{-\phi}}{8} F^{PQ} F_{PQ} U^2 - g^2 e^{\phi} U^2 = 0, \tag{38}$$

$$-\left(\frac{\partial_m U}{U}\right)_{,0} + 4\left(\frac{\partial_0 U}{U}\right)_{,m} + \partial_0\phi\partial_m\phi = 0, \tag{39}$$

$$\left(\frac{\partial_0 U}{U}\right)_{,0} + 2\left(\frac{\partial_0 U}{U}\right)^2 - \frac{1}{4U^2}D_m\left(\gamma^{mn}\partial_n U^4\right) + \frac{e^{-\phi}}{8}F^{PQ}F_{PQ}U^2 - g^2 e^{\phi}U^2 = 0, \tag{40}$$

$$4D_n\left(\frac{\partial_m U}{U}\right) + 4\frac{\partial_m U}{U}\frac{\partial_n U}{U} - R^{m'}{}_{mm'n}$$
$$+\partial_m\phi\partial_n\phi + e^{-\phi}\left(F_m{}^P F_{Pn} - \frac{1}{8}F^{PQ}F_{PQ}g_{mn}\right) + g^2 e^{\phi}g_{mn} = 0, \tag{41}$$

$$-\frac{1}{U^4}\partial_0\left(U^2\partial_0\frac{\phi}{2}\right) + \frac{1}{U^4}D_m\left(U^4\gamma^{mn}\partial_n\frac{\phi}{2}\right) + \frac{e^{-\phi}}{8}F^{MN}F_{MN} - g^2 e^{\phi} = 0. \tag{42}$$

If we again assume the form $U = a(t)W(x^m)$, then the $(0-m)$ component of the Einstein equation implies $\phi = \phi(t)$. In addition, from the ϕ equation of motion, using the solution for the flux $F \propto U(t,x^m)^{-4}$, we find that $W = 1$ because each term has a different dependence on W and the ϕ equation of motion can not be satisfied if U depends on x^m. Hence we must have $U = a(t)$ and moreover since the spacetime no longer has a non-trivial warp factor $W(x^m)$, it can not be warped. Given the above result, the equations of motion now can be written as:

$$3\left(\frac{\partial_0 a}{a}\right)_{,0} + \partial_0\phi\partial_0\phi + \frac{e^{-\phi}}{8}F^{PQ}F_{PQ}a^2 - g^2 e^{\phi}a^2 = 0, \tag{43}$$

$$\left(\frac{\partial_0 a}{a}\right)_{,0} + 2\left(\frac{\partial_0 a}{a}\right)^2 + \frac{e^{-\phi}}{8}F^{PQ}F_{PQ}a^2 - g^2 e^{\phi}a^2 = 0, \tag{44}$$

$$-\frac{1}{a^2}\partial_0\left(a^2\partial_0\frac{\phi}{2}\right) + \frac{e^{-\phi}}{8}F^{MN}F_{MN}a^2 - g^2 e^{\phi}a^2 = 0, \tag{45}$$

$$\frac{3}{4}e^{-\phi}F^{PQ}F_{PQ} + 2g^2 e^{\phi} = g^{mn}R^{m'}{}_{mm'n}(x^m) \equiv R_c(= \text{const}). \tag{46}$$

Notice that the term $g^{mn}R^{m'}{}_{mm'n}$ could in principle be a function of x^m, but in this case it is not allowed by Eq. (46) as the left hand side depends only on t. This fact means that the compact extra space must be a constant curvature two dimensional sphere. Here there is no way to introduce branes which induce a deficit angle and deform a sphere with a constant curvature into a rugby ball shape. Therefore, we can see that this ansatz, namely varying U and ϕ with static r can not lead to satisfactory solutions.

C. Time dependent r and ϕ, but static U.

Finally, let us try to obtain a solution of the static three space with a dynamical extra dimension. If we take $r = r(t)$ and $\phi = \phi(t, x^m)$, but $U = W(x^m)$, then Einstein's equations and the equation of motion for ϕ are

$$2\left[\left(\frac{\partial_0 r}{r}\right)_{,0} + \left(\frac{\partial_0 r}{r}\right)^2\right] + \partial_0\phi\partial_0\phi - \frac{D_m(\gamma^{mn}\partial_n W^4)}{4r^2W^4} + \frac{e^{-\phi}}{8}F^{PQ}F_{PQ} - g^2e^{\phi} = 0, \quad (47)$$

$$-4\frac{\partial_m W}{W}\frac{\partial_0 r}{r} + \partial_0\phi\partial_m\phi = 0, \quad (48)$$

$$\frac{D_m(\gamma^{mn}\partial_n W^4)}{4r^2W^4} - \frac{e^{-\phi}}{8}F^{PQ}F_{PQ} + g^2e^{\phi} = 0, \quad (49)$$

$$4D_n\left(\frac{\partial_m W}{W}\right) + 4\frac{\partial_m W}{W}\frac{\partial_n W}{W} - R^{m'}{}_{mm'n} - \frac{1}{2U^2}\left(\partial_0 r^2\right)_{,0}\gamma_{mn}$$
$$+\partial_m\phi\partial_n\phi + e^{-\phi}\left(F_m{}^PF_{Pn} - \frac{1}{8}F^{PQ}F_{PQ}g_{mn}\right) + g^2e^{\phi}g_{mn} = 0, \quad (50)$$

$$-\frac{1}{W^2r^2}\partial_0\left(r^2\partial_0\frac{\phi}{2}\right) + \frac{1}{W^4r^2}D_m\left(W^4\gamma^{mn}\partial_n\frac{\phi}{2}\right) + \frac{e^{-\phi}}{8}F^{MN}F_{MN} - g^2e^{\phi} = 0. \quad (51)$$

Now provided that $\phi(t, x^m)$ can be decomposed as $\phi(t, x^m) = \phi(t) + \phi(x^m)$ and F depends only on x^m, then Eqs. (47) and (49) can be reduced to

$$r(t)^2e^{\phi(t)} = 1, \quad (52)$$

$$\left(\frac{\partial_0 r}{r}\right)_{,0} + \left(\frac{\partial_0 r}{r}\right)^2 + \frac{1}{2}\partial_0\phi\partial_0\phi = 0, \quad (53)$$

$$\frac{D_m(\gamma^{mn}\partial_n W^4)}{4W^4} - \frac{e^{-\phi(x^m)}}{8}F^{PQ}F_{PQ} + g^2e^{\phi(x^m)} = 0. \quad (54)$$

Unfortunately, the solution of these equations are not compatible with

$$\partial_0 r^2 = 0, \quad (55)$$

which can be obtained from Eq. (51) using Eq. (52). Thus, we once again see that there is no consistent solution with this ansatz.

IV. TIME DEPENDENT SOLUTIONS WITH DYMANICAL r, U AND ϕ

Having tried unsuccessfully to obtain static solutions for r and U, we now look for dynamical solutions where all the key fields r, U and ϕ are time dependent. We again make

a series of ansatz, in this case $\phi(t,x^m)=\phi(t)+\phi(x^m)$, and assume the separable form of $U=a(t)W(x^m)$. The Einstein equations and the equation of motion for ϕ are:

$$\frac{3}{U^2}\left(\frac{\partial_0 a}{a}\right)_{,0}+\frac{2}{U^2}\left[\left(\frac{\partial_0 r}{r}\right)_{,0}-\frac{\partial_0 r}{r}\frac{\partial_0 a}{a}+\left(\frac{\partial_0 r}{r}\right)^2\right]+\frac{\partial_0\phi\partial_0\phi}{U^2}$$
$$-\frac{1}{4r^2W^4}D_m\left(\gamma^{mn}\partial_n W^4\right)+\frac{e^{-\phi}}{8}F^{PQ}F_{PQ}-g^2e^{\phi}=0, \tag{56}$$
$$-4\frac{\partial_m W}{W}\frac{\partial_0 r}{r}+\partial_0\phi\partial_m\phi=0, \tag{57}$$
$$\frac{1}{U^2}\left(\frac{\partial_0 a}{a}\right)_{,0}+\frac{2}{U^2}\left(\frac{\partial_0 a}{a}\right)^2+\frac{2}{U^2}\frac{\partial_0 a}{a}\frac{\partial_0 r}{r}$$
$$-\frac{1}{4r^2W^4}D_m(\gamma^{mn}\partial_n W^4)+\frac{e^{-\phi}}{8}F^{PQ}F_{PQ}-g^2e^{\phi}=0, \tag{58}$$
$$4D_n\left(\frac{\partial_m W}{W}\right)+4\frac{\partial_m W}{W}\frac{\partial_n W}{W}-R^{m'}{}_{mm'n}-\left[\left(\frac{\partial_0 r^2}{2W^2}\right)_{,0}+4\frac{\partial_0 a}{a}\frac{\partial_0 r}{r}\frac{r^2}{U^2}\right]\gamma_{mn}$$
$$+\partial_m\phi\partial_n\phi+e^{-\phi}\left(F_m{}^PF_{Pn}-\frac{1}{8}F^{PQ}F_{PQ}g_{mn}\right)+g^2e^{\phi}g_{mn}=0, \tag{59}$$
$$-\frac{1}{U^4r^2}\partial_0\left(r^2U^2\partial_0\frac{\phi}{2}\right)+\frac{1}{W^4r^2}D_m\left(W^4\gamma^{mn}\partial_n\frac{\phi}{2}\right)+\frac{e^{-\phi}}{8}F^{MN}F_{MN}-g^2e^{\phi}=0. \tag{60}$$

Under the additional ansatz that $e^{\phi(t)}r^2=1$ (motivated by the observation that $r^2\propto e^{-\phi}$ [3]), which is equivalent to

$$\partial_0\phi=-2\frac{\partial_0 r}{r}\,, \tag{61}$$

we see that Eq. (57), leads to

$$\partial_m\phi=-2\frac{\partial_m W}{W}, \tag{62}$$

as in the GGP solution. If we further assume that the field strength F is static and only depends on x^m, then the field equations Eqs. (56), (58) and (60) coupled with Eq. (62) can be rewritten as the following differential equation which depends only on x^m,

$$C(x^m)-\frac{D_m(\gamma^{mn}\partial_n W^4)}{4W^4}+\frac{e^{-\phi}r^2}{8}F^{PQ}F_{PQ}-g^2e^{\phi}r^2=0, \tag{63}$$

where, $C(x^m)$ is given by

$$\begin{aligned} C(x^m) &= \frac{r^2}{U^2}\left[3\left(\frac{\partial_0 a}{a}\right)_{,0}+2\left(\frac{\partial_0 r}{r}\right)_{,0}-2\frac{\partial_0 r}{r}\frac{\partial_0 a}{a}+2\left(\frac{\partial_0 r}{r}\right)^2+\partial_0\phi\partial_0\phi\right] \\ &= \frac{r^2}{U^2}\left[\left(\frac{\partial_0 a}{a}\right)_{,0}+2\left(\frac{\partial_0 a}{a}\right)^2+2\frac{\partial_0 a}{a}\frac{\partial_0 r}{r}\right] \\ &= -\frac{1}{U^4}\partial_0\left(r^2U^2\partial_0\frac{\phi}{2}\right), \end{aligned} \tag{64}$$

each equality in Eq. (64) arising from Eqs. (56), (58) and (60), respectively.

A. Power law solutions for r and a

Eqns. (63) and (64) still look very difficult to solve directly from first principles, and so instead we will try to obtain solutions by assuming the form of r and a, and looking for self-consistency in the solutions. As a first attempt we assume power law behaviour for them, namely:

$$a \propto t^n, \quad r \propto t^{n_r}. \tag{65}$$

The three equalities in Eq. (64) coupled with Eq. (61) now become

$$\begin{aligned} C(x^m) &= \frac{t^{2(n_r-n-1)}}{W^2}(-3n-2n_r-2n_r n+6n_r^2) \\ &= \frac{t^{2(n_r-n-1)}}{W^2}n_r(2n+2n_r-1) \\ &= \frac{t^{2(n_r-n-1)}}{W^2}n(2n+2n_r-1). \end{aligned} \tag{66}$$

There are two possible ways in which $C(x^m)$ can be a function of only x^m, as required by Eq. (63). The first is if the time dependent prefactor vanishes which corresponds to $n_r - n - 1 = 0$. The second way is if the right hand side of each of the terms vanish, which corresponds to the brackets vanishing in Eq. (66). The former is precisely the structure found by Tolley et al [12] based on a scaling argument for the scale factors. However, the metric ansatz of $g_{\mu\nu}$ in [12] is slightly different to ours (11). In particular it follows that the condition $n_r - n - 1 = 0$ is not a solution in our case, because it can not satisfy all three equalities in Eqs. (66). In fact we determine the values of n and n_s in Eqs. (66) by equating the coefficients :

$$-3n-2n_r-2n_r n+6n_r^2 = n_r(2n+2n_r-1) = n(2n+2n_r-1). \tag{67}$$

This has the non trivial solution

$$n = \frac{2\pm\sqrt{3}}{4}, \quad n_r = \mp\frac{\sqrt{3}}{4}. \tag{68}$$

This in turn gives $C(x^m) = 0$ which is consistent with the above discussions and is compatible with Eq. (59) too, because the solution satisfies

$$\left(\frac{\partial_0 r^2}{2a^2}\right)_{,0} + 4\frac{\partial_0 a}{a}\frac{\partial_0 r}{r}\frac{r^2}{a^2} = 0. \tag{69}$$

Notice that in this case, we obtain identical solutions for F, ϕ and W as found in the GGP solution. This is as expected, since the x^m dependent part of the field equations are identical to that of the GGP solution. In this sense we have obtained the time dependent version of the GGP solution.

From the metric ansatz Eqn. (11) it follows that the time t is actually the conformal time in the usual sense. The "cosmic time" τ can therefore be defined as $d\tau \propto t^n dt$, from which we obtain

$$\begin{aligned}
&ds^2 = W(x^m)^2[-d\tau^2 + a(\tau)^2\delta_{ij}dx^i dx^j] + r(\tau)^2 ds_2^2, \\
&a(\tau) \propto \tau^{n/(n+1)}, \quad r(\tau) \propto \tau^{n_r/(n+1)},
\end{aligned} \tag{70}$$

in terms of the cosmic time.

B. Exponential solutions for r and a

The next obvious step is to assume an exponential form

$$a(t) = e^{ht}, \quad r(t) = e^{h_r t}, \tag{71}$$

where h and h_r are constants. In this case, Eq. (64) with Eq. (61) leads to

$$\begin{aligned}
C(x^m) &= \frac{e^{2(h_r-h)t}}{W^2} 2h_r(-h+3h_r) \\
&= \frac{e^{2(h_r-h)t}}{W^2} 2h_r(h+h_r) \\
&= \frac{e^{2(h_r-h)t}}{W^2} 2h(h+h_r).
\end{aligned} \tag{72}$$

which now has a non trivial solution $h = h_r$. Then, $C(x^m)$ is given by

$$C(x^m) = \frac{4h^2}{W(x^m)^2}. \tag{73}$$

Thus, we obtain the equations of motion of the x^m dependent part of the fields to be

$$\frac{4h^2}{W(x^m)^2} - \frac{D_m(\gamma^{mn}\partial_n W^4)}{4W^4} + \frac{e^{-\phi(x^m)}}{8} F^{PQ}F_{PQ} - g^2 e^{\phi(x^m)} = 0, \tag{74}$$

$$\frac{4h^2}{W(x^m)^2}\gamma_{mn} - \partial_m\phi\partial_n\phi - e^{-\phi(x^m)}\left(F_m{}^P F_{Pn} - \frac{1}{8}F^{PQ}F_{PQ}\gamma_{mn}\right) - g^2 e^{\phi(x^m)}\gamma_{mn} = 0. \tag{75}$$

Something significant can now be seen. Recall that we have equation (62), relating ϕ and $W(x^m)$. Given the solution we have just obtained, we see that in Eqn. (75), by introducing

$\tilde{g}^2 \equiv g^2 - 4h^2$, then a new solution to the system is obtained which looks identical to the original x^m part of the GGP solution but with our redefined gauge coupling $\tilde{g}^2$ replacing the original g^2 coupling. Obviously, the $h \to 0$ limit corresponds to the original static GGP solution. Hence, we have obtained the explicit expression of the solution including the x^m dependent part. However, notice that since h is just a constant it could in principle take any value. In particular, for $4h^2 > g^2$, corresponding to a negative $\tilde{g}^2$, we find that the x^m dependent part of the solution has only differs slightly from that obtained in GGP. We show this and give the actual solution for the case of vanishing and negative $\tilde{g}^2$ in Appendix B. The line element of this solution with such a nonvanishing h is rewritten as

$$ds^2 = W(x^m)^2[-d\tau^2 + (h\tau)^2\delta_{ij}dx^i dx^j] + (h\tau)^2 ds_2^2, \tag{76}$$

in terms of the cosmic time. This solution is the same as that found in Ref. [12], however, the x^m dependence of the solution was not explicitely solved for there. Here, we have shown that it is same as that of the GGP solution.

V. CONCLUSIONS

We have derived a new class of exact time dependent solutions in a six dimensional gauged supergravity compactified on a two dimensional axisymmetric space. Under the assumption of a separable form of U we showed that there is no solution expressing the either an expanding four dimensional universe with a static internal space or visa versa. Exact solutions we obtained involved all the dimensions either expanding or contracting which means the eventual decompactification or collapse of the extra dimension, indicating an instability of Salam-Sezgin, $(\text{Minkowski})_4 \times S^2$, spacetime for the case with the absence of the maximal symmetry in the four dimensional spacetime.

In the above analysis, we did not include into the action brane terms such as

$$S_{\text{brane}} = \sum_i \int d^4x\, T_i = \sum_i \int d^6x\, T_i \delta^{(2)}(x^m - x_i^m)\,, \tag{77}$$

where T_i is the tension of the 'i-th' brane and x_i^m denotes the position of the brane in the internal space. However, we can easily introduce such brane terms, their affect being to induce the deficit angle in the internal space. The topological condition for the gauge field A_M is the same as we previously obtained for the static solutions, because the solution of

the gauge field strength F_{mn} is unchanged in the presence of the branes. As is discussed in [4], for the case of $r_0^2 \neq r_1^2$ in Eq. (24), while one pole can be smooth, the other has a deficit angle

$$\frac{\delta}{2\pi} = 1 - \frac{r_1^2}{r_0^2}. \tag{78}$$

Combining Eq. (25) and the topological condition, the Dirac quantization condition, for the gauge field A_M becomes [4, 15],

$$\frac{4g}{q} = N, \tag{79}$$

leading to the quantized deficit angle

$$\frac{\delta}{2\pi} = 1 - N^2, \tag{80}$$

which was previously obtained for the static solution with N being an integer [4]. However, for the new solution in Section IV B, Eq. (78) is rewritten as

$$\frac{\delta}{2\pi} = 1 - \frac{8\tilde{g}^2}{q^2}, \tag{81}$$

for a positive $\tilde{g}^2$,

$$\frac{\delta}{2\pi} = 1, \tag{82}$$

for a vanishing $\tilde{g}^2$ and,

$$\frac{\delta}{2\pi} = 1 - \frac{8(-\tilde{g}^2)}{q^2}, \tag{83}$$

for a negative $\tilde{g}^2$. Hence the deficit angle is given as

$$\frac{\delta}{2\pi} = 1 - N^2 \left| 1 - 4\frac{h^2}{g^2} \right| . \tag{84}$$

This implies that the interval of the quantized deficit angle becomes narrow for $2h \approx g$ in the time-dependent solution. It would be interesting to investigate the consequence of this new deficit angle.

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APPENDIX A: CONVENTIONS AND GEOMETRICAL QUANTITIES

Here, we note our conventions and several geometrical quantities.

Christoffel symbols

$$\Gamma^M_{NP} = \frac{1}{2} g^{MQ} (g_{QN,P} + g_{QP,N} - g_{NP,Q}). \tag{A1}$$

Riemann tensor

$$R^M{}_{NOP} = \Gamma^M_{NP,O} - \Gamma^M_{NO,P} + \Gamma^M_{QO}\Gamma^Q_{NP} - \Gamma^M_{QP}\Gamma^Q_{NO}. \tag{A2}$$

With this definition, the sign in front of the Einstein term in the action is a plus.

Metric

$$ds^2 = U^2(t, x^m)(-dt^2 + \delta_{ij} dx^i dx^j) + r(t)^2 \gamma(x^m)_{mn} dx^m dx^n, \tag{A3}$$

where $i, j, ...,$ run over the usual three-spatial dimensions and $m, n, ...,$ run over the extra spatial dimensions.

Christoffel symbols

$$\begin{array}{ll}
\Gamma^0_{00} = \frac{1}{2}\frac{\partial_0 U^2}{U^2} & \Gamma^0_{0m} = \frac{1}{2}\frac{\partial_m U^2}{U^2} \\
\Gamma^0_{ij} = \frac{1}{2}\frac{\partial_0 U^2}{U^2}\delta_{ij} & \Gamma^0_{mn} = \frac{1}{2}\frac{\partial_0 r^2}{U^2}\gamma_{mn} \\
\Gamma^i_{0j} = \frac{1}{2}\frac{\partial_0 U^2}{U^2}\delta^i{}_j & \Gamma^i_{jm} = \frac{1}{2}\frac{\partial_m U^2}{U^2}\delta^i{}_j \\
\Gamma^m_{00} = \frac{1}{2}\frac{\partial_n U^2}{r^2}\gamma^{mn} & \Gamma^m_{0n} = \frac{1}{2}\frac{\partial_0 r^2}{r^2}\delta^m{}_n \\
\Gamma^m_{ij} = -\frac{1}{2}\frac{\partial_n U^2}{r^2}\gamma^{mn}\delta_{ij} & \Gamma^m_{np} = \frac{1}{2}\gamma^{mq}(\gamma_{qn,p} + \gamma_{qp,n} - \gamma_{np,q}) \\
\text{others} = 0 &
\end{array}$$

Ricci tensors

$$\begin{aligned}
R_{00} &= -\delta^i{}_i \left(\frac{\partial_0 U}{U}\right)_{,0} + \frac{1}{r^2}\gamma^{mn}\partial_m U \partial_n U (\delta^i{}_i - 1) \\
&\quad + \left[-\left(\frac{\partial_0 r}{r}\right)_{,0} + \frac{\partial_0 r}{r}\frac{\partial_0 U}{U} - \left(\frac{\partial_0 r}{r}\right)^2\right]\delta^n{}_n + \frac{D_m\left(\gamma^{mn}\partial_n U^2\right)}{2r^2}, && \text{(A4)} \\
R_{0m} &= \left(\frac{\partial_m U}{U}\right)_{,0} - (\delta^i{}_i + 1)\left(\frac{\partial_0 U}{U}\right)_{,m} + \frac{\partial_m U}{U}\frac{\partial_0 r}{r}(\delta^i{}_i + \delta^n{}_n - 1), && \text{(A5)} \\
R_{ij} &= \left[\left(\frac{\partial_0 U}{U}\right)_{,0} + \left(\frac{\partial_0 U}{U}\right)^2 (\delta^k{}_k - 1) - \frac{\partial_m U}{U}\frac{\partial_n U}{U}\frac{U^2}{r^2}\gamma^{mn}(\delta^k{}_k - 1)\right. \\
&\quad \left. + \frac{\partial_0 U}{U}\frac{\partial_0 r}{r}\delta^m{}_m - \frac{D_m(\gamma^{mn}\partial_n U^2)}{2r^2}\right]\delta_{ij}, && \text{(A6)}
\end{aligned}$$

$$
\begin{aligned}
R_{mn} &= -(\delta^i{}_i+1)D_n\left(\frac{\partial_m U}{U}\right)-(\delta^i{}_i+1)\frac{\partial_m U}{U}\frac{\partial_n U}{U}+R^{m'}{}_{mm'n} \\
&\quad+\left[\left(\frac{\partial_0 r^2}{2U^2}\right)_{,0}+\frac{\partial_0 U}{U}\frac{\partial_0 r}{r}\frac{r^2}{U^2}(1+\delta^i{}_i)+\left(\frac{\partial_0 r}{r}\right)^2\frac{r^2}{U^2}(\delta^{m'}{}_{m'}-2)\right]\gamma_{mn}, \quad \text{(A7)} \\
\text{others} &= 0. \quad \text{(A8)}
\end{aligned}
$$

APPENDIX B: SOLUTIONS FOR VANISHING AND NEGATIVE $\tilde{g}^2$

In this Appendix, we note the solutions for the case of vanishing or negative $\tilde{g}^2$. Following GGP, by introducing the variables

$$
\begin{aligned}
x &= \frac{1}{2}\phi+\ln A, \\
y &= \frac{1}{2}\phi+4\ln W+\ln A, \\
z &= -\phi-2\ln W,
\end{aligned} \quad \text{(B1)}
$$

we obtain

$$
\left(\frac{dy}{d\eta}\right)^2+4g^2e^{2y}=\lambda_2^2, \quad \text{(B2)}
$$

and similar equations for x and z, both of which are decoupled from y [4]. Here, λ_2 is a constant for the first integral and η is a coordinate in the coordinate system

$$
ds_2^2=W^8A^2d\eta^2+A^2d\psi^2. \quad \text{(B3)}
$$

The solution for a positive g^2 is presented in Ref. [4] as

$$
y=-\ln\cosh(\lambda_2(\eta-\eta_2))+\frac{1}{2}\ln\left(\frac{\lambda_2^2}{(4g^2)}\right). \quad \text{(B4)}
$$

However, as one can see, in Sec. IV B, the effective g^2, namely $\tilde{g}^2$, can be positive or negative in some time dependent solutions. This then means that the solution for y is replaced with

$$
y=\lambda_2(\eta-\eta_2), \quad \text{(B5)}
$$

for the case of vanishing $\tilde{g}^2$ and

$$
y=-\ln\sinh(\lambda_2(\eta-\eta_2))+\frac{1}{2}\ln\left(\frac{\lambda_2^2}{(-4\tilde{g}^2)}\right), \quad \text{(B6)}
$$

for a negative $\tilde{g}^2$.

We then obtain

$$AW^4 = \begin{cases} \frac{1}{(\cosh^3(\lambda_1(\eta-\eta_1))\cosh(\lambda_2(\eta-\eta_2)))^{1/4}} \frac{\cosh(\lambda_1(\eta-\eta_1))}{\cosh(\lambda_2(\eta-\eta_2))} \left(\frac{\lambda_2^2}{4\tilde{g}^2}\right)\left(\frac{q^2}{2\lambda_1^2}\right)^{-1} & \text{for positive } \tilde{g}^2 \\ \frac{1}{(\cosh^3(\lambda_1(\eta-\eta_1))\sinh(\lambda_2(\eta-\eta_2)))^{1/4}} \frac{\cosh(\lambda_1(\eta-\eta_1))}{\sinh(\lambda_2(\eta-\eta_2))} \left(\frac{\lambda_2^2}{-4\tilde{g}^2}\right)\left(\frac{q^2}{2\lambda_1^2}\right)^{-1} & \text{for negative } \tilde{g}^2 \\ \frac{1}{(\cosh^3(\lambda_1(\eta-\eta_1))e^{-\lambda_2(\eta-\eta_2)})^{1/4}} \frac{\cosh(\lambda_1(\eta-\eta_1))}{e^{\lambda_2(\eta-\eta_2)}} \left(\frac{q^2}{2\lambda_1^2}\right)^{-1} & \text{for vanishing } \tilde{g}^2 \end{cases} . \tag{B7}$$

and

$$A = \begin{cases} \frac{1}{(\cosh^3(\lambda_1(\eta-\eta_1))\cosh(\lambda_2(\eta-\eta_2)))^{1/4}} \left(\frac{\lambda_2^2}{4\tilde{g}^2}\right)^{1/2}\left(\frac{q^2}{2\lambda_1^2}\right)^{-3/2} & \text{for positive } \tilde{g}^2 \\ \frac{1}{(\cosh^3(\lambda_1(\eta-\eta_1))\sinh(\lambda_2(\eta-\eta_2)))^{1/4}} \left(\frac{\lambda_2^2}{-4\tilde{g}^2}\right)^{1/2}\left(\frac{q^2}{2\lambda_1^2}\right)^{-3/2} & \text{for negative } \tilde{g}^2 \\ \frac{1}{(\cosh^3(\lambda_1(\eta-\eta_1))e^{-\lambda_2(\eta-\eta_2)})^{1/4}} \left(\frac{q^2}{2\lambda_1^2}\right)^{-3/2} & \text{for vanishing } \tilde{g}^2 \end{cases} . \tag{B8}$$

Here, λ_1 is a constant for the first integral with repect with x, and we used the solution of x given in [4]. Setting $\lambda_1 = \lambda_2 = 1$ and introducing a new coordinate $dr = AW^4 d\eta$, allows us to derive the deficit angle given in Eqs. (81), (82) and (83).

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